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Possibility mean and variation coefficient based ranking methods for type-1 fuzzy numbers and interval type-2 fuzzy numbers

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Abstract.Although there are many ranking methods for type-1 fuzzy numbers, most of which exist some limitations. In this paper, we propose a new concept to rank type-1 fuzzy numbers, which defines the formats of possibility mean and variation coefficient. It not only clearly discriminates the ranking of the type-1 fuzzy numbers especially for the reasonable ranking of symmetric type-1 fuzzy numbers, but also satisfies the logicality of the ranking with their images'. Some properties of the proposed ranking methods are discussed. Then, we extend the concept to interval type-2 fuzzy numbers environment, and present a new ranking method with possibility mean and variation coefficient forms. Some properties are also discussed. For ranking the type-1 fuzzy numbers and interval type-2 fuzzy numbers, the proposed ranking methods are easy to understand and their computations are simple. Several examples are proposed to illustrate the ranking of type-1 fuzzy numbers and interval type-2 fuzzy numbers, and the results show that it enriches the existing fuzzy ranking methods.

Keywords: Fuzzy ranking, possibility mean, variation coefficient, type-1 fuzzy numbers, interval type-2 fuzzy numbers

1. Introduction

The fuzzy set theory introduced by Zadeh [35] has achieved a great success in various fields. Later, Zadeh [36] introduced the ordinary fuzzy set named type-1 fuzzy set (T1 FSs) and the type-2 fuzzy sets (T2 FSs). Because of the high computational complexity in using T2 FSs, Mendel [20] further defined the notion of interval type-2 fuzzy sets (IT2 FSs), which is a typical form of T2 FSs. T1 FSs and IT2 FSs have been found useful to deal with vagueness and uncertainty in decision problems [18, 19, 23, 25]. However, ranking fuzzy numbers is one of the most important step for decision making under uncertainty environment [9, 17, 21, 31].

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For type-1 fuzzy numbers, we divide the existing ranking methods into four categories. (1) The first category is centroid methods. Yagger [33] firstly proposed the centroid index point to order type-1 fuzzy numbers. Lee and Li [16] proposed the mean and standard deviation values to distinguish type-1 fuzzy numbers, but the comparison criteria is not clear. Cheng [10] tried to improve Yager's [33] and Lee and Li's [16] approaches, but the ranking result is inconsistent with peoples' intuition. Chu and Tsao [11] defined the area to order type-1 fuzzy numbers. Wang and Lee [29] used the importance degrees to revise the Chu and Tsao's method [11], but the ranking result is inconsistent with peoples' intuition. Ali Duzce [13] presented a new method for ranking type-1 trapezial fuzzy numbers using nine-point center. (2) The second category is maximizing and minimizing set methods. Chen [5] introduced the maximizing and minimizing sets, but the ranking order would

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be changed when inserting or deleting type-1 fuzzy numbers. Deng [12] presented an ideal solution to rank type-1 fuzzy numbers. (3) The third category is L-R deviation degree methods. Wang et al. [30] defined the L-R deviation degree. Asady [2] used the transfer coefficient to revise Wang et al.'s [30] method, but the ranking of their images' is not logical. Nejad and Mashinchi [22] redefined the L-R deviation degree, but the ranking of their images' is not logical either. Yu et al. [34] introduced epsilon deviation degree to rank symmetric type-1 fuzzy numbers, but the computation is too complexity. (4) The last category is distance methods. Asady and Zendehnam [4] introduced the minimization distance, but it could not discriminate the symmetric type-1 fuzzy numbers with different spans in the bottom. Abbasbandy and Hajjari [1] proposed the magnitude method to revise Asady and Zendehnam's [4] method, but it still could not solve the problem. Ezzati et al. [14] revised the magnitude method, but the ranking of type-1 fuzzy numbers' images' is not logical. Saneifard and Allahviranloo [24] used regular weighted function to rank type-1 fuzzy numbers, but the ranking of their images' is still not logical. Janizade-Haji et al. [15] developed a distance method using $\alpha-$ cut for ranking generalized type-1 fuzzy numbers. Chen et al. [8] proposed a method for standardized generalized type-1 fuzzy numbers with different left height and right heights.

Compared with the ranking methods for type-1 fuzzy numbers, the methods for interval type-2 fuzzy numbers are far less. Mitchell [21] used the weighted standard deviation to measure the uncertainties of interval type-2 fuzzy numbers. Lee and Chen [7, 17] employed the mean value and deviation of the vertex points to order interval type-2 fuzzy numbers. Wu and Mendel [32] introduced the centroid method, and discussed some properties of it. Qin and Liu [23] proposed combined ranking method to order interval type-2 fuzzy numbers. Singh [25] presented a new family of utmost distance measures to rank type-2 fuzzy numbers.

From the analysis above, it is concluded that the current type-1 fuzzy ranking methods exist the following problems. Firstly, the ranking is not consistent with people's intuition [10, 29]. Secondly, the ranking of the fuzzy numbers' images' is not logical [11, 14, 22]. Thirdly, the symmetric fuzzy numbers having different spans in the bottom could not be discriminated [1, 4]. To solve these problems, the paper proposes the new concepts to rank type-1 fuzzy numbers, that is the combination of possibility mean and variation coefficient, which avoids the problems that most existing ranking methods have. Meanwhile, the proposed ranking method also satisfies the general ranking principles of the type-1 fuzzy numbers. Moreover, some properties are proposed and discussed. Then, the new concept is extended to rank interval type-2 fuzzy numbers, which is not the simple superposition of the lower membership function result and the upper membership function result. They are not only discriminate the interval type-2 fuzzy numbers especially for the symmetric interval type-2 fuzzy numbers with different spreads in the bottom, but also the ranking of their images'. Some properties of the proposed ranking method for interval type-2 fuzzy numbers are also discussed.

This paper is organized as follows: Section 2 introduces the concepts of type-1 fuzzy numbers, interval type-2 fuzzy numbers and the related distance based ranking methods for type-1 fuzzy numbers. Section 3 introduces the main problems of the existing ranking methods for type-1 fuzzy numbers, and then proposes a new ranking method. Some properties are discussed as well. Section 4 proposes a ranking method for interval type-2 fuzzy numbers, and the properties are also discussed. Section 5 illustrates some examples to order type-1 fuzzy numbers and interval type-2 fuzzy numbers, and compares the results with those of some existing ranking methods. Section 6 summarizes the main results and draws conclusions.

2. Preliminaries

In this section, we introduce the concepts of type-1 fuzzy numbers, interval type-2 fuzzy numbers and the related distance based ranking methods for type-1 fuzzy numbers.

2.1. The concepts of type-1 fuzzy numbers and interval type-2 fuzzy numbers

Definition 1. [28] A type-1 fuzzy number $\tilde{A}$ is a pair $(\underline{A}, \overline{A})$ of functions $\underline{A}(r)$, $\overline{A}(r)$, which satisfies the following conditions:

(1) $\underline{A}(r)$ is a bounded increasing continuous function,
(2) $\overline{A}(r)$ is a bounded decreasing continuous function,
(3) $\underline{A}(r) \leq \overline{A}(r)$, $0 \leq r \leq 1$.

The type-1 fuzzy number is a trapezoidal fuzzy number $\tilde{A} = (x_0, y_0, \alpha, \beta)$ with two defuzzifiers

x_0, y_0, the left fuzziness α and the right fuzziness β, with which the membership function is defined as follows.

$$\underline{A}(r) = x_0 - \alpha + \alpha r, \quad \overline{A}(r) = y_0 + \beta - \beta r.$$

For the support set of $\tilde{A}(S(\tilde{A}))$ is defined as:

$$S(\tilde{A}) = \{r|\tilde{A}(r) > 0\} = [\underline{A}(r), \overline{A}(r)].$$

Definition 2. [20] An interval type-2 fuzzy number $\tilde{\tilde{A}}$ is an objective, which has the parametric form as follows.

$$\tilde{\tilde{A}} = \int_{x\in X}\left[\int_{u\in J_x} 1/(x,u)\right]/x, \qquad (1)$$

where x is the primary variable, $J_x \in [0,1]$ is the primary membership of x, u is the secondary variable, and $\int_{u\in J_x} 1/(x,u)$ is the secondary membership function at x.

The uncertainty footprint of $\tilde{\tilde{A}}(FOU(\tilde{\tilde{A}}))$ is defined as follows.

$$FOU(\tilde{\tilde{A}}) = \bigcup_{x\in X} J_x,$$
$$= \{(x,y) : y \in J_x = [\tilde{A}^U(x), \tilde{A}^L(x)]\},$$

where FOU is shown as the shaded region. It is bounded by an upper membership function (UMF) $\tilde{A}^U(x)$ and a lower membership function (LMF) $\tilde{A}^L(x)$, both of which are type-1 fuzzy numbers. An example of interval type-2 fuzzy number is shown in Fig. 1.

Wu and Mendel [32] discussed the properties of the ordering methods for interval type-2 fuzzy numbers, which are denoted as follows.

Property 1. *For an arbitrary finite subset* Γ *of set* E.

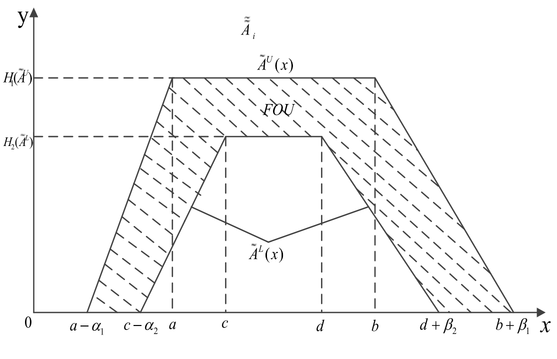


Fig. 1. The sample of interval type-2 fuzzy number.

(1) If $\tilde{\tilde{A}} \succeq \tilde{\tilde{B}}$ and $\tilde{\tilde{B}} \succeq \tilde{\tilde{A}}$, then $\tilde{\tilde{A}} \sim \tilde{\tilde{B}}$.
(2) If $\tilde{\tilde{A}} \succeq \tilde{\tilde{B}}$ and $\tilde{\tilde{B}} \succeq \tilde{\tilde{C}}$, then $\tilde{\tilde{A}} \succeq \tilde{\tilde{C}}$.
(3) If $\tilde{\tilde{A}} \cap \tilde{\tilde{B}} = \emptyset$ and $\tilde{\tilde{A}}$ is on the right of $\tilde{\tilde{B}}$, then $\tilde{\tilde{A}} \succeq \tilde{\tilde{B}}$.
(4) The order of $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is not affected by other interval type-2 fuzzy numbers under comparison.
(5) If $\tilde{\tilde{A}} \succeq \tilde{\tilde{B}}$, then $\tilde{\tilde{A}} + \tilde{\tilde{C}} \succeq \tilde{\tilde{B}} + \tilde{\tilde{C}}$.
(6) If $\tilde{\tilde{A}} \succeq \tilde{\tilde{B}}$, then $\tilde{\tilde{A}}\tilde{\tilde{C}} \succeq \tilde{\tilde{B}}\tilde{\tilde{C}}$.

Where $\succeq$ means "larger than or equal to" in the sense of ranking, $\sim$ means "the same rank", $\cap$ means the overlap of the two fuzzy numbers.

It is denoted that the properties are also applied to type-1 fuzzy numbers [26, 27], but not all the existing ranking methods satisfy the above properties at the same time.

2.2. *The related distance based ranking methods for type-1 fuzzy numbers*

Definition 3. [4] For a trapezoidal fuzzy number $\tilde{A} = (x_0, y_0, \alpha, \beta)$ with parametric form $\tilde{A} = (\underline{A}(r), \overline{A}(r))$, the magnitude of which is defined as:

$$Mag(A) = \frac{1}{2}\left(\int_0^1 (\underline{A}(r) + \overline{A}(r))dr\right), \qquad (2)$$

where $\underline{A}(r) = x_0 - \alpha + \alpha r$, $\quad \overline{A}(r) = y_0 + \beta - \beta r$.

Definition 4. [3] For a trapezoidal fuzzy number $\tilde{A} = (x_0, y_0, \alpha, \beta)_{(L,R)}$ with parametric form $\tilde{A} = (\underline{A}_\epsilon(r), \overline{A}_\epsilon(r))$, the magnitude of which is redefined as:

$$Mag(\tilde{A})_\epsilon = \frac{1}{2}\int_0^1 \left(\underline{A}_\epsilon(r) + \overline{A}_\epsilon(r)\right) dr,$$
$$= \frac{(x_0 + y_0) + (\beta - \alpha) + \frac{\alpha}{1+L}\left(L + \epsilon^{\frac{1+L}{L}}\right)}{2},$$
$$+ \frac{-\frac{\beta}{1+R}\left(R + \epsilon^{\frac{1+R}{R}}\right)}{2} \qquad (3)$$

where $L, R > 0$, $A_\epsilon = (\underline{A}_\epsilon(r), \overline{A}_\epsilon(r))$ is the best approximate epsilon-neighborhood of fuzzy number $\tilde{A}$.

For two type-1 fuzzy numbers $\tilde{A}$ and $\tilde{B}$, the ranking criteria based on the Asady's method [3] is defined as follows.

(1) If $Mag(\tilde{A}) > Mag(\tilde{B})$, then $\tilde{A} \succ \tilde{B}$;
(2) If $Mag(\tilde{A}) < Mag(\tilde{B})$, then $\tilde{A} \prec \tilde{B}$;

(3) If $Mag(\tilde{A}) = Mag(\tilde{B})$, then
   (a) if $Mag(\tilde{A}_\epsilon) > Mag(\tilde{B}_\epsilon)$, then $\tilde{A} \succ \tilde{B}$;
   (b) if $Mag(\tilde{A}_\epsilon) < Mag(\tilde{B}_\epsilon)$, then $\tilde{A} \prec \tilde{B}$;
   (c) else $\tilde{A} \sim \tilde{B}$.

It is denoted that $\succ$ means "larger than" in the sense of ranking, $\prec$ means "smaller than" in the sense of ranking, $\sim$ means "the same rank".

3. The new ranking methods for type-1 fuzzy numbers

3.1. *The main problems of existing ranking methods for type-1 fuzzy numbers*

We give several examples to illustrate the main problems of the existing ranking methods for type-1 fuzzy numbers. Here, we just list several ranking methods to present the problems, more ranking methods will be shown in comparison with the proposed ranking method.

Problem 1. The ranking is not consistent with peoples' intuition, which is shown in Example 1.

Example 1. Consider the following sets of type-1 fuzzy numbers, which are shown in Fig. 2.

Set 1: $\tilde{A} = (1, 13, 1)$, $\tilde{B} = (\frac{1}{12}, 2, 1)$ and $\tilde{C} = (0, 1, 6, 0)$ in Ref. [3].
Set 2: $\tilde{A} = (3, 1, 5)$, $\tilde{B} = (3, 7, 1, 1)$ and $\tilde{C} = (3, 1, 7)$ in Ref. [22].

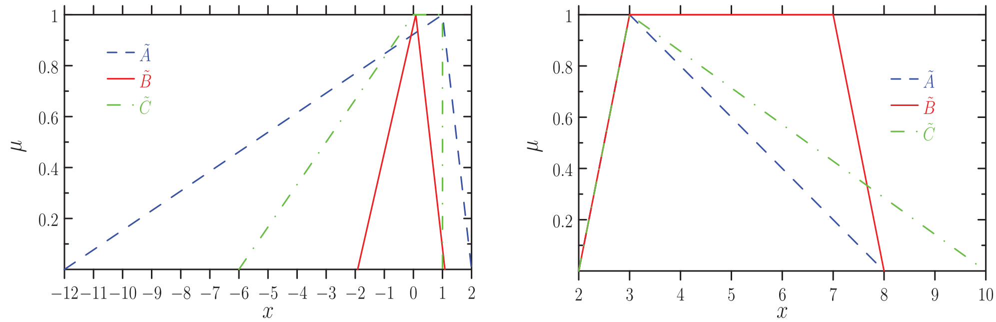
In Set 1, which is shown in Fig. 2(a), the ranking order for Wang and Lee's method [29] is $\tilde{C} \succ \tilde{B} \succ \tilde{A}$, which is not consistent with our intuition.

In Set 2, which is shown in Fig. 2(b), the ranking order for Wang et al.'s method [30] is worthless. As the transfer coefficient $\lambda_{\tilde{B}} = \lambda_{\tilde{C}} = 0$, which leads to the left deviation degree $d^L_{\tilde{B}}$ and $d^L_{\tilde{C}}$ are worthless.

Problem 2. The ranking of the type-1 fuzzy numbers' images' are not logical, which is shown in Example 2.

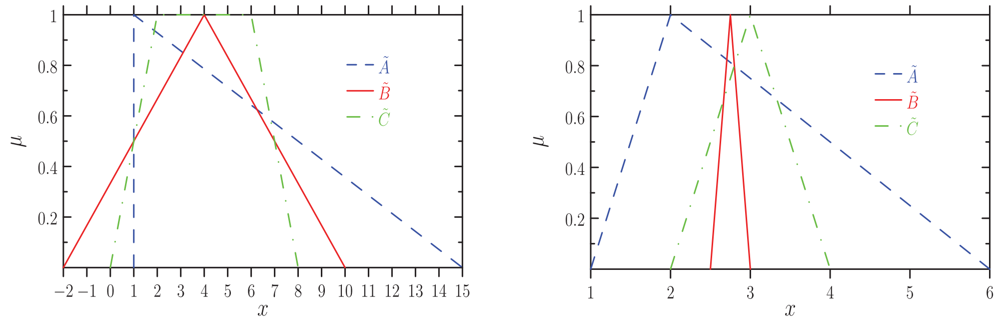
Example 2. Consider the following sets of type-1 fuzzy numbers, which are shown in Fig. 3.

Set 3: $\tilde{A} = (1, 0, 14)$, $\tilde{B} = (4, 6, 6)$ and $\tilde{C} = (2, 6, 2, 2)$ in Ref. [14].



(a) Set 1 (b) Set 2

Fig. 2. The type-1 fuzzy numbers of Example 1.



(a) Set 3 (b) Set 4

Fig. 3. The type-1 fuzzy numbers of Example 2.

Set 4: $\tilde{A} = (2, 1, 4)$, $\tilde{B} = (2.75, 0.25, 0.25)$ and $\tilde{C} = (3, 1, 1)$ in Ref. [34].

In Set 3, the ranking order for Ezzati et al.'s method [14] is $\tilde{A} \succ \tilde{C} \succ \tilde{B}$, but the ranking of their images' is $-\tilde{C} \succ -\tilde{B} \succ -\tilde{A}$, which is not logical.

In Set 4, the ranking order for Nejad and Mashinchi's method [22] is $\tilde{C} \succ \tilde{B} \succ \tilde{A}$, but the ranking of their images' is $-\tilde{B} \succ -\tilde{A} \succ -\tilde{C}$, which is not logical either.

Problem 3. The ranking of the symmetric type-1 fuzzy numbers having different spreads in the bottom can not be discriminated, which is shown in Example 3.

Example 3. Consider the two type-1 fuzzy numbers $\tilde{A} = (0.5, 0.3, 0.3)$, $\tilde{B} = (0.5, 0.1, 0.1)$, which are shown in Fig. 4.

The ranking for Abbasbandy and Hajjari's method [1] and Asady's method [3] is $\tilde{A} \sim \tilde{B}$, which is not logical, as they actually present different fuzzy information.

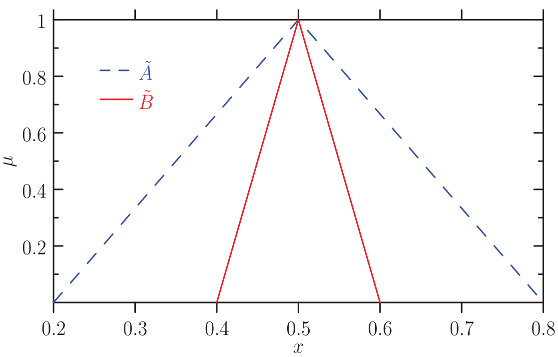


Fig. 4. The type-1 fuzzy numbers of Example 3.

3.2. *The proposed ranking method for type-1 fuzzy numbers*

To overcome the ranking problems shown above, we introduce a new concept to rank type-1 fuzzy numbers, that is the combination of possibility mean and variation coefficient. For the proposed ranking method, the variation coefficient is a new format, but the possibility mean is originated from Ref. [1], which are defined as follows.

Definition 5. [1] For any type-1 fuzzy number $\tilde{A} = (x_0, y_0, \alpha, \beta)$ with parametric form $\tilde{A} = (\underline{A}(r), \overline{A}(r))$, the possibility mean of which is defined as:

$$M(A) = \frac{1}{2}\int_0^1 \left(\underline{A}(r) + \overline{A}(r) + x_0 + y_0\right) f(r)dr, \tag{4}$$

where $f(r)$ is an increasing function satisfying $f(0) = 0$, $f(1) = 1$, $\int_0^1 f(r)dr = \frac{1}{2}$,

$$\underline{A}(r) = x_0 - \alpha + \alpha r, \quad \overline{A}(r) = y_0 + \beta - \beta r. \tag{5}$$

Definition 6. For any type-1 fuzzy number $\tilde{A} = (x_0, y_0, \alpha, \beta)$, the variation coefficient of which is defined as:

$$VC(\tilde{A}) = \begin{cases} \dfrac{V(\tilde{A})}{M(\tilde{A})}, & \text{if} \quad M(\tilde{A}) \neq 0, \\[2ex] \dfrac{V(\tilde{A})}{\epsilon}, & \text{if} \quad M(\tilde{A}) = 0, \end{cases} \tag{6}$$

where ϵ is an extremely small value to present the approximate $M(\tilde{A})$, $V(\tilde{A})$ is the deviation value of the type-1 fuzzy number, and the expression of which is defined as:

$$V(\tilde{A}) = \frac{1}{4}\int_0^1 \left(\overline{A}(r) + y_0 - \underline{A}(r) - x_0\right)^2 f(r)dr, \tag{7}$$

where $f(r)$ is an increasing function satisfying $f(0) = 0$, $f(1) = 1$ and $\int_0^1 f(r)dr = \frac{1}{2}$.

It is denoted that ϵ can be seen as a extremely small positive or negative number. For example, if the possibility mean value of type-1 fuzzy number $\tilde{A}$ is approximated to zero from the positive direction and the corresponding approximating value is ϵ, then the image's ranking of $\tilde{A}$ $(-\tilde{A})$ must be approximated to zero from the negative direction, and the corresponding approximating value is $-\epsilon$.

Next, we introduce the new ordering rules for the proposed ranking method. For any two type-1 fuzzy numbers $\tilde{A}$ and $\tilde{B}$, the comparison criteria is carried out as below.

Definition 7. Let $\tilde{A}$ and $\tilde{B}$ be two type-1 fuzzy numbers,

(1) If $M(\tilde{A}) > M(\tilde{B})$, then $\tilde{A} \succ \tilde{B}$;
(2) If $M(\tilde{A}) < M(\tilde{B})$, then $\tilde{A} \prec \tilde{B}$;
(3) If $M(\tilde{A}) = M(\tilde{B})$, then
  (a) if $VC(\tilde{A}) > VC(\tilde{B})$, then $\tilde{A} \succ \tilde{B}$;
  (b) if $VC(\tilde{A}) < VC(\tilde{B})$, then $\tilde{A} \prec \tilde{B}$;
  (c) else $\tilde{A} \sim \tilde{B}$.

It is denoted that $\succ$ means "larger than" in the sense of ranking, $\prec$ means "smaller than" in the sense of ranking, $\sim$ means "the same rank".

Regarding the proposed ranking method for type-1 fuzzy numbers, the possibility mean value represents information from the membership degree, variation coefficient reflects the change rate of span length from the right side to the left side. The combination of both not only compares the information of type-1 fuzzy numbers, but also discriminates them from having the same possibility mean value but with different variation rates. Hence, we use the combination of both to rank the type-1 fuzzy numbers. We firstly rank $\tilde{A}$ and $\tilde{B}$ based on their possibility mean values if the two values are different. Otherwise, we further compare the variation coefficient to identify their rankings.

In the following, we study some properties of the proposed ranking method for type-1 fuzzy numbers.

Theorem 1. *For the type-1 fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$,*

(1) If $\tilde{A} \succeq \tilde{B}$ and $\tilde{B} \succeq \tilde{A}$, then $\tilde{A} \sim \tilde{B}$.
(2) If $\tilde{A} \succeq \tilde{B}$ and $\tilde{B} \succeq \tilde{C}$, then $\tilde{A} \succeq \tilde{C}$.
(3) If $\tilde{A} \cap \tilde{B} = \emptyset$ and $\tilde{A}$ is on the right of $\tilde{B}$, then $\tilde{A} \succeq \tilde{B}$.
(4) The order of $\tilde{A}$ and $\tilde{B}$ is not affected by other type-1 fuzzy numbers under comparison.
(5) If $\tilde{A} \succeq \tilde{B}$, then $\tilde{A} + \tilde{C} \succeq \tilde{B} + \tilde{C}$.

Where $\succeq$ means "larger than or equal to" in the sense of ranking, $\sim$ means "the same rank", $\cap$ means the overlap of the two fuzzy numbers.

Proof. The proof of Theorem 1 (1)-(4) are easy, we here give the proof process of Theorem 1 (5) in detail. (5) Suppose $\tilde{A} = (x_a, y_a, \alpha_a, \beta_a)$, $\tilde{B} = (x_b, y_b, \alpha_b, \beta_b)$ and $\tilde{C} = (x_c, y_c, \alpha_c, \beta_c)$.

As we use the combination of possibility mean and variation coefficient to rank type-1 fuzzy numbers, the comparison criteria can also be divided into two cases.

Case 1: $\tilde{A} \succeq \tilde{B}$ if and only if $M(\tilde{A}) \succeq M(\tilde{B})$.

From Equation (4), the difference of possibility mean values for $\tilde{A} + \tilde{C}$ and $\tilde{B} + \tilde{C}$ can be written as:

$$\begin{aligned} &M(\tilde{A}+\tilde{C}) - M(\tilde{B}+\tilde{C}) \\ &= \frac{1}{2}\int_0^1 \big(\underline{A}(r) + \overline{A}(r) + x_a + y_a \\ &\quad -\underline{B}(r) - \overline{B}(r) - x_b - y_b\big)\, f(r)dr. \end{aligned} \tag{8}$$

Because of $M(\tilde{A}) \succeq M(\tilde{B})$, from Equation (4), it is right that $M(A+C) - M(B+C) \geq 0$. That is

$$\tilde{A} + \tilde{C} \succeq \tilde{B} + \tilde{C}.$$

Case 2: $\tilde{A} \succeq \tilde{B}$ if and only if $M(\tilde{A}) = M(\tilde{B})$ and $VC(\tilde{A}) \succeq VC(\tilde{B})$.

Because of $M(\tilde{A}) = M(\tilde{B})$, it is right that $M(\tilde{A} + \tilde{C}) = M(\tilde{B} + \tilde{C})$.

From Equation (7), it is obvious that

$$\begin{aligned} &V(\tilde{A}+\tilde{C}) - V(\tilde{B}+\tilde{C}) \\ &= \frac{1}{4}\int_0^1 \big(\overline{A}(r) + y_a - \underline{A}(r) - x_a + \overline{B}(r) + y_b \\ &\quad - \underline{B}(r) - x_b + 2\big(\overline{C}(r) + y_c - \underline{C}(r) - x_c\big)\big) \\ &\quad \big(\overline{A}(r) + y_a - \underline{A}(r) - x_a - \big(\overline{B}(r) + y_b - \\ &\quad \underline{B}(r) - x_b\big)\big)\, f(r)dr. \end{aligned} \tag{9}$$

As $M(\tilde{A}) = M(\tilde{B})$ and $VC(\tilde{A}) \succeq VC(\tilde{B})$, from Equation (6), it is concluded that $V(\tilde{A}) \geq V(\tilde{B})$.

That is $\overline{A}(r) + y_a - \underline{A}(r) - x_a + \overline{B}(r) + y_b - \underline{B}(r) - x_b \geq 0$.

In Equation (9), as $(\overline{A}(r) + y_a - \underline{A}(r) - x_a + \overline{B}(r) + y_b - \underline{B}(r) - x_b + 2(\overline{C}(r) + y_c - \underline{C}(r) - x_c)) \geq 0$.

Combined with the conclusions above, it is right that $V(\tilde{A}+\tilde{C}) - V(\tilde{B}+\tilde{C}) \geq 0$. That is

$$\tilde{A} + \tilde{C} \succeq \tilde{B} + \tilde{C}.$$

The proof of Theorem 1(5) is completed.

The proof of Theorem 1 is completed. □

Besides, there are some other properties that the proposed ranking method has, such as multiplication and the logicality of the ranking with their images'.

Theorem 2. *For the type-1 fuzzy numbers $\tilde{A}$, if λ is a real number, then $M(\lambda\tilde{A}) = \lambda M(\tilde{A})$ and $VC(\lambda\tilde{A}) = \lambda VC(\tilde{A})$.*

Proof. In Ref. [6], it is concluded that $\lambda\tilde{A} = (\lambda x_a, \lambda y_a, \lambda\alpha_a, \lambda\beta_a)$. Combined with Equation (4), it can easily be seen that

$$\begin{aligned} M(\lambda\tilde{A}) &= \frac{\lambda}{2}\int_0^1 \big(\underline{A}(r) + \overline{A}(r) + x_0 + y_0\big)\, f(r)dr, \\ &= \lambda M(\tilde{A}). \end{aligned}$$

From Equation (7), it is also right that

$$\begin{aligned} V(\lambda\tilde{A}) &= \frac{\lambda^2}{4}\int_0^1 \big(\overline{A}(r) + y_0 - \underline{A}(r) - x_0\big)^2 f(r)dr, \\ &= \lambda^2 V(\tilde{A}). \end{aligned}$$

Such that from Equation (6), it is concluded that

$$VC(\lambda\tilde{A}) = \frac{V(\lambda\tilde{A})}{M(\lambda\tilde{A})} = \lambda VC(\tilde{A}).$$

The proof of Theorem 2 is completed. □

Theorem 3. *For the type-1 fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$, if $\tilde{A} \succeq \tilde{B} \succeq \tilde{C}$, then $-\tilde{C} \succeq -\tilde{B} \succeq -\tilde{A}$, where $\succeq$ means "larger than or equal to" in the sense of ranking.*

Proof. Suppose $\tilde{A} = (x_a, y_a, \sigma_a, \beta_a)$, $\tilde{B} = (x_b, y_b, \sigma_b, \beta_b)$ and $\tilde{C} = (x_c, y_c, \sigma_c, \beta_c)$.

As we use the combination of the possibility mean and variation coefficient to rank type-1 fuzzy numbers, there are four cases of the comparison criteria to rank the type-1 fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$.

Case 1: $\tilde{A} \succeq \tilde{B} \succeq \tilde{C}$ if and only if $M(\tilde{A}) \succeq M(\tilde{B}) \succeq M(\tilde{C})$.

From the results of Theorem 2, it is concluded that $M(-\tilde{A}) = -M(\tilde{A})$, $M(-\tilde{B}) = -M(\tilde{B})$ and $M(-\tilde{C}) = -M(\tilde{C})$. Coupled with $M(\tilde{A}) \succeq M(\tilde{B}) \succeq M(\tilde{C})$, it is also right that $-M(\tilde{C}) \succeq -M(\tilde{B}) \succeq -M(\tilde{A})$.

That is $-\tilde{C} \succeq -\tilde{B} \succeq -\tilde{A}$.

Case 2: $\tilde{A} \succeq \tilde{B} \succeq \tilde{C}$ if and only if $M(\tilde{A}) = M(\tilde{B}) \succeq M(\tilde{C})$ and $VC(\tilde{A}) \succeq VC(\tilde{B})$.

From the results of Theorem 2, it is concluded that $M(-\tilde{A}) = -M(\tilde{A})$, $M(-\tilde{B}) = -M(\tilde{B})$, $M(-\tilde{C}) = -M(\tilde{C})$, $VC(-\tilde{A}) = -VC(\tilde{A})$ and $VC(-\tilde{B}) = -VC(\tilde{B})$.

As $M(\tilde{B}) \succeq M(\tilde{C})$, it is clear that $-\tilde{C} \succeq -\tilde{B}$.

Coupled with $VC(\tilde{A}) \succeq VC(\tilde{B})$, it is also right that $-VC(\tilde{B}) \succeq -VC(\tilde{A})$ and $-\tilde{B} \succeq -\tilde{A}$.

That is $-\tilde{C} \succeq -\tilde{B} \succeq -\tilde{A}$.

Case 3: $\tilde{A} \succeq \tilde{B} \succeq \tilde{C}$ if and only if $M(\tilde{A}) \succeq M(\tilde{B}) = M(\tilde{C})$ and $VC(\tilde{B}) \succeq VC(\tilde{C})$. The proof principle is similar to that of Theorem 2 Case 2, we omit it here.

Case 4: $\tilde{A} \succeq \tilde{B} \succeq \tilde{C}$ if and only if $M(\tilde{A}) = M(\tilde{B}) = M(\tilde{C})$ and $VC(\tilde{A}) \succeq VC(\tilde{B}) \succeq VC(\tilde{C})$.

From the results of Theorem 2, it is obvious that $VC(-\tilde{A}) = -VC(\tilde{A})$, $VC(-\tilde{B}) = -VC(\tilde{B})$, $VC(-\tilde{C}) = -VC(\tilde{C})$. As $M(\tilde{A}) = M(\tilde{B}) = M(\tilde{C})$ and $VC(\tilde{A}) \succeq VC(\tilde{B}) \succeq VC(\tilde{C})$, it is right that $-VC(\tilde{C}) \succeq -VC(\tilde{B}) \succeq -VC(\tilde{A})$.

That is $-C \succeq -B \succeq -A$.

The proof of Theorem 3 is completed. □

4. The ranking method for interval type-2 fuzzy numbers

In this section, we extend the concepts of possibility mean and variation coefficient to rank interval type-2 fuzzy numbers, which is not the simple superposition of the mean value for lower membership function and upper membership function. Detailed definitions and expressions can be found in this section.

Definition 8. For an arbitrary interval type-2 fuzzy number $\tilde{\tilde{A}} = [(a, b, \alpha_1, \beta_1; H_1(\tilde{A}^U)), (c, d, \alpha_2, \beta_2; H_2(\tilde{A}^L))]$, the possibility mean of which is defined as:

$$M(\tilde{\tilde{A}}) = \frac{M(\tilde{A}^U) + M(\tilde{A}^L)}{2}, \tag{10}$$

where the possibility mean values of the lower membership function and upper membership function are written as:

$$M(\tilde{A}^U) = \frac{1}{2}\int_0^{H_1(\tilde{A}^U)} (\underline{A}^U(r) + \overline{A}^U(r) + a + b)\, f(r)dr, \tag{11}$$

$$M(\tilde{A}^L) = \frac{1}{2}\int_0^{H_2(\tilde{A}^L)} (\underline{A}^L(r) + \overline{A}^L(r) + c + d)\, f(r)dr, \tag{12}$$

$f(r)$ is an increasing function satisfying $f(0) = 0$, $f(1) = 1$ and $\int_0^1 f(r)dr = \frac{1}{2}$.

Definition 9. For an arbitrary interval type-2 fuzzy number $\tilde{\tilde{A}} = [(a, b, \alpha_1, \beta_1; H_1(\tilde{A}^U)), (c, d, \alpha_2, \beta_2; H_2(\tilde{A}^L))]$, the variation coefficient of which is defined as:

$$VC(\tilde{\tilde{A}}) = \begin{cases} \dfrac{V(\tilde{\tilde{A}})}{M(\tilde{\tilde{A}})}, & \text{if} \quad M(\tilde{\tilde{A}}) \neq 0, \\ \dfrac{V(\tilde{\tilde{A}})}{\epsilon}, & \text{if} \quad M(\tilde{\tilde{A}}) = 0. \end{cases} \tag{13}$$

It is denoted that ϵ is an extremely small value to present the approximate $M(\tilde{\tilde{A}})$, $V(\tilde{\tilde{A}})$ is the variation value. The expression of variation $V(\tilde{\tilde{A}})$ is defined as:

$$V(\tilde{\tilde{A}}) = \sqrt{V(\tilde{A}^U)V(\tilde{A}^L)}, \tag{14}$$

and

$$V(\tilde{A}^U) = \frac{1}{4}\int_0^{H_1(\tilde{A}^U)} (\overline{A}^U(r) + b - \underline{A}^U(r) - a)^2 f(r)dr, \quad (15)$$

$$V(\tilde{A}^L) = \frac{1}{4}\int_0^{H_2(\tilde{A}^L)} (\overline{A}^L(r) + d - \underline{A}^L(r) - c)^2 f(r)dr, \quad (16)$$

where $f(r)$ is an increasing function satisfying $f(0) = 0$, $f(1) = 1$ and $\int_0^1 f(r)dr = \frac{1}{2}$.

It is denoted that ϵ can be denoted as a extremely small positive or negative number, for example, if the possibility mean value of interval type-2 fuzzy number $\tilde{\tilde{A}}$ is approximated to zero from the positive direction and the corresponding approximating is ϵ, then the image of $\tilde{\tilde{A}}$ $(-\tilde{\tilde{A}})$ must be approximated to zero from the negative direction, and the corresponding approximating ought to be $-\epsilon$.

Next, we will introduce the new ordering rules for the proposed ranking method. For any two interval type-2 fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$, the comparison criteria is carried out as follows.

Definition 10. Let $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ be two interval type-2 fuzzy numbers.

(1) If $M(\tilde{\tilde{A}}) > M(\tilde{\tilde{B}})$, then $\tilde{\tilde{A}} \succ \tilde{\tilde{B}}$;
(2) If $M(\tilde{\tilde{A}}) < M(\tilde{\tilde{B}})$, then $\tilde{\tilde{A}} \prec \tilde{\tilde{B}}$;
(3) If $M(\tilde{\tilde{A}}) = M(\tilde{\tilde{B}})$, then
  (a) if $VC(\tilde{\tilde{A}}) > VC(\tilde{\tilde{B}})$, then $\tilde{\tilde{A}} \succ \tilde{\tilde{B}}$;
  (b) if $VC(\tilde{\tilde{A}}) < VC(\tilde{\tilde{B}})$, then $\tilde{\tilde{A}} \prec \tilde{\tilde{B}}$;
  (c) else $\tilde{\tilde{A}} \sim \tilde{\tilde{B}}$.

It is denoted that $\succ$ means "larger than" in the sense of ranking, $\prec$ means "smaller than" in the sense of ranking, $\sim$ means "the same rank".

Regarding the proposed ranking method for interval type-2 fuzzy numbers, the possibility mean represents the information from the lower membership degree to upper membership degree, variation coefficient reflects the change rate of span length from the right side to the left side. The combination of which not only compares the information of the interval type-2 fuzzy numbers, but also discriminates the interval type-2 fuzzy numbers from having the same possibility mean but with different variation rates. Hence, we will use the combination of both to rank the interval type-2 fuzzy numbers. We firstly rank $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ based on their possibility mean values $M(\tilde{\tilde{A}})$ and $M(\tilde{\tilde{B}})$ if the two values are different. Otherwise, we further compare the variation coefficients $VC(\tilde{\tilde{A}})$ and $VC(\tilde{\tilde{B}})$ to identify their orders.

In the following, we study some properties of the proposed ranking method for interval type-2 fuzzy numbers.

Theorem 4. *For the interval type-2 fuzzy numbers $\tilde{\tilde{A}}$, $\tilde{\tilde{B}}$ and $\tilde{\tilde{C}}$, there are some properties as follows.*

(1) If $\tilde{\tilde{A}} \succeq \tilde{\tilde{B}}$ and $\tilde{\tilde{B}} \succeq \tilde{\tilde{A}}$, then $\tilde{\tilde{A}} \sim \tilde{\tilde{B}}$.
(2) If $\tilde{\tilde{A}} \succeq \tilde{\tilde{B}}$ and $\tilde{\tilde{B}} \succeq \tilde{\tilde{C}}$, then $\tilde{\tilde{A}} \succeq \tilde{\tilde{C}}$.
(3) If $\tilde{\tilde{A}} \cap \tilde{\tilde{B}} = \emptyset$ and $\tilde{\tilde{A}}$ is on the right of $\tilde{\tilde{B}}$, then $\tilde{\tilde{A}} \succeq \tilde{\tilde{B}}$.
(4) The order of $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is not affected by other interval type-2 fuzzy numbers under comparison.

Where $\succeq$ means "larger than or equal to" in the sense of ranking, $\sim$ means "the same rank", $\cap$ means the overlap of two fuzzy numbers.

Proof. The proof of Theorem 4 (1)-(4) are easy, we omit here. □

Remark 1. In view of the computing complexity of the existing ranking methods for interval type-2 fuzzy numbers, the proposed ranking method just satisfies the conclusions of Property 1 (1)-(4).

Besides, there are some other properties that the proposed ranking method have, such as multiplication and the ranking with their images'.

Theorem 5. *For the interval type-2 fuzzy numbers $\tilde{\tilde{A}}$, if λ is a real number, then $M(\lambda\tilde{\tilde{A}}) = \lambda M(\tilde{\tilde{A}})$ and $VC(\lambda\tilde{\tilde{A}}) = \lambda VC(\tilde{\tilde{A}})$.*

Proof. Suppose interval type-2 fuzzy number $\tilde{\tilde{A}} = [(a, b, \alpha_1, \beta_1; H_1(x_1)), (c, d, \alpha_2, \beta_2; H_2(x_2))]$.

In Ref. [7], it is concluded that $\lambda\tilde{\tilde{A}} = [(\lambda a, \lambda b, \lambda\alpha_1, \lambda\beta_1; H_1(x_1)), (\lambda c, \lambda d, \lambda\alpha_2, \lambda\beta_2; H_2(x_2))]$. Coupled with Equations (12) and (13), it is obvious that

$$\begin{aligned} M(\lambda\tilde{\tilde{A}}) &= \frac{M(\lambda\tilde{A}^U) + M(\lambda\tilde{A}^L)}{2}, \\ &= \frac{\lambda M(\tilde{A}^U) + \lambda M(\tilde{A}^L)}{2}, \\ &= \lambda M(\tilde{\tilde{A}}) \end{aligned} \quad (17)$$

From Equations (16) and (17), it is also right that

$$V(\lambda\tilde{\tilde{A}}) = \sqrt{\lambda^2 V(\tilde{A}^U)\lambda^2 V(\tilde{A}^L)}.$$
$$= \lambda^2\sqrt{V(\tilde{A}^U)V(\tilde{A}^L)}. \tag{18}$$

Once again, substitute Equations (17) and (18) into Equation (13), it is clear that

$$VC(\lambda\tilde{\tilde{A}}) = \frac{V(\lambda\tilde{\tilde{A}})}{M(\lambda\tilde{\tilde{A}})} = \frac{\lambda^2 V(\tilde{\tilde{A}})}{\lambda M(\tilde{\tilde{A}})}$$
$$= \lambda VC(\tilde{\tilde{A}}).$$

The proof of Theorem 5 is completed. □

Theorem 6. *For the interval type-2 fuzzy numbers $\tilde{\tilde{A}}$, $\tilde{\tilde{B}}$ and $\tilde{\tilde{C}}$, if $\tilde{\tilde{A}} \succeq \tilde{\tilde{B}} \succeq \tilde{\tilde{C}}$, then $-\tilde{\tilde{C}} \succeq -\tilde{\tilde{B}} \succeq -\tilde{\tilde{A}}$.*

Proof. The proof principle is similar to that of Theorem 3, we omit it here. □

5. Examples

5.1. The ranking for type-1 fuzzy numbers

To present the meaning of the proposed ranking method with possibility mean and variation coefficient, it is used to solve the three kinds of problems in Section 3.1. An additional example is listed to present the property of Theorem 1 (5).

Example 1. Consider the three type-1 fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ of Set 1 shown in Fig. 2(a), using the proposed ranking method, the ranking of the three type-1 fuzzy numbers and their images is $\tilde{B} \succ \tilde{C} \succ \tilde{A}$ and $-\tilde{A} \succ -\tilde{C} \succ -\tilde{B}$, respectively.

Using the same method, the three type-1 fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ of Set 2 shown in Fig. 2(b) and their images is ranked as $\tilde{B} \succ \tilde{C} \succ \tilde{A}$ and $-\tilde{A} \succ -\tilde{C} \succ -\tilde{B}$, respectively.

Table 1 lists the ranking results with some existing ranking methods and the proposed ranking method, it can easily be seen that the proposed ranking method is able to overcome the condition that the ranking is not consistent with peoples' intuition and the limitation of Wang et al.'s method [30].

Example 2. Consider the three type-1 fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ of Set 3 shown in Fig. 3(a), using the proposed method, the ranking of the three type-1 fuzzy numbers and their images is $\tilde{C} \succ \tilde{B} \succ \tilde{A}$ and $-\tilde{A} \succ -\tilde{B} \succ -\tilde{C}$, respectively.

Using the same method, the three type-1 fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ of Set 4 shown in Fig. 3(b)

Table 1
The comparative results of Example 1

Authors	Set	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$	Ranking
Wang and Lee (2008) [29]	1	-3	-0.25	-1.375	$\tilde{B} \succ \tilde{C} \succ \tilde{A}$
	2	4.33	(5, 0.5)	(5, 0.96)	$\tilde{C} \succ \tilde{B} \succ \tilde{A}$
Wang et al. (2009) [30]	1	0	0.05	8.4	$\tilde{C} \succ \tilde{B} \succ \tilde{A}$
	2	0	0.44	0.44	$\tilde{C} \sim \tilde{B} \succ \tilde{A}$
Asady (2010) [2]	1	2.22	4.44	3.14	$\tilde{B} \succ \tilde{C} \succ \tilde{A}$
	2	0.31	0.55	0.42	$\tilde{B} \succ \tilde{C} \succ \tilde{A}$
Asady (2011) [3]	1	-2	-0.17	-1.33	$\tilde{B} \succ \tilde{C} \succ \tilde{A}$
	2	4	5	4.5	$\tilde{B} \succ \tilde{C} \succ \tilde{A}$
The proposed method	1	$(0, -4.08/\epsilon)$	$(0, -0.19/\epsilon)$	$(0, -2.25/\epsilon)$	$\tilde{B} \succ \tilde{C} \succ \tilde{A}$
	2	3.33	5	3.5	$\tilde{B} \succ \tilde{C} \succ \tilde{A}$

Table 2
The comparative results of Example 2

Authors	Set	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$	Ranking	$-\tilde{A}$	$-\tilde{B}$	$-\tilde{C}$	Ranking
Asady (2010) [2]	3	12	0	0	$\tilde{A} \succ \tilde{B} \sim \tilde{C}$	0	22	4.75	$-\tilde{B} \succ -\tilde{C} \succ -\tilde{A}$
	4	0.47	0	0.57	$\tilde{C} \succ \tilde{A} \succ \tilde{B}$	0	1.44	1.2	$-\tilde{B} \succ -\tilde{C} \succ -\tilde{A}$
Nejad and Mashinchi	3	0.18	0.22	0.28	$\tilde{C} \succ \tilde{B} \succ \tilde{A}$	1.41	1.65	1.28	$-\tilde{B} \succ -\tilde{A} \succ -\tilde{C}$
(2011) [22]	4	2	0	1.5	$\tilde{A} \succ \tilde{C} \succ \tilde{B}$	0	3.13	0	$-\tilde{B} \succ -\tilde{A} \sim -\tilde{C}$
Ezzati et al. (2012)	3	2.17	$4+6\sigma$	$4+4\sigma$	$\tilde{B} \succ \tilde{C} \succ \tilde{A}$	-2.17	$-4+6\sigma$	$-4+4\sigma$	$-\tilde{A} \succ -\tilde{B} \succ -\tilde{C}$
[14]	4	2.25	2.75	3	$\tilde{C} \succ \tilde{B} \succ \tilde{A}$	-2.25	-2.75	-3	$-\tilde{A} \succ -\tilde{B} \succ -\tilde{C}$
viSaneifard and Allah-	3	1.17	1.22	1.24	$\tilde{C} \succ \tilde{B} \succ \tilde{A}$	0.71	0.73	0.72	$-\tilde{B} \succ -\tilde{C} \succ -\tilde{A}$
ranloo(2012)[24]	4	0.71	0.6	0.65	$\tilde{A} \succ \tilde{C} \succ \tilde{B}$	0.53	0.49	0.45	$-\tilde{A} \succ -\tilde{B} \succ -\tilde{C}$
The proposed method	3	2.17	4	3.33	$\tilde{B} \succ \tilde{C} \succ \tilde{A}$	-2.17	-4	-3.33	$-\tilde{A} \succ -\tilde{C} \succ -\tilde{B}$
	4	2.25	2.75	3	$\tilde{C} \succ \tilde{B} \succ \tilde{A}$	-2.25	-2.75	-3	$-\tilde{A} \succ -\tilde{B} \succ -\tilde{C}$

and their images is ranked as $\tilde{C} \succ \tilde{B} \succ \tilde{A}$ and $-\tilde{A} \succ -\tilde{B} \succ -\tilde{C}$, respectively.

Table 2 lists the ranking results with some existing methods and the proposed method. From the comparative ranking results, it is clear that the proposed method is able to overcome the defect that their images' ranking is not logical.

Example 3. Consider the two type-1 fuzzy numbers $\tilde{A}$ and $\tilde{B}$ shown in Figure 4, using the proposed method, the ranking of the two type-1 fuzzy numbers and their images is $\tilde{A} \succ \tilde{B}$ and $-\tilde{B} \succ -\tilde{A}$, respectively.

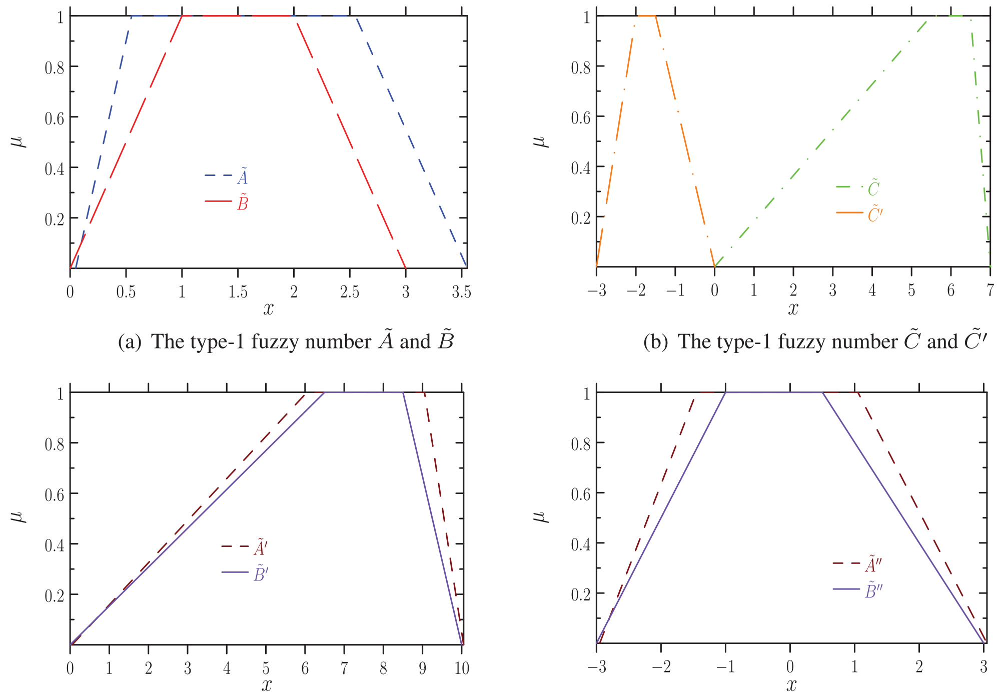
Table 3 lists the ranking results with the proposed method. From the comparative ranking results, it is obvious that the proposed method is able to overcome the defeat that the ranking of the symmetric type-1 fuzzy numbers is not logical.

Example 4 demonstrates the property of Theorem 1 (5).

Example 4. Consider the two type-1 fuzzy numbers $\tilde{A}$ and $\tilde{B}$, $\tilde{C}$ and $\tilde{C}'$ are two arbitrary type-1 fuzzy numbers shown in Fig. 5. Let $\tilde{A}' = \tilde{A} + \tilde{C}$, $\tilde{B}' = \tilde{B} + \tilde{C}$, $\tilde{A}'' = \tilde{A} + \tilde{C}'$, $\tilde{B}'' = \tilde{B} + \tilde{C}'$.

Table 3
The comparative results of Example 3

Authors	Attitude	$\tilde{A}$	$\tilde{B}$	Ranking	$-\tilde{A}$	$-\tilde{B}$	Ranking
Asady (2011) [3]		0.5	0.5	$\tilde{A} \sim \tilde{B}$	-0.5	-0.5	$-\tilde{A} \sim -\tilde{B}$
Ezzati et al. (2012) [14]		0.65	0.55	$\tilde{A} \succ \tilde{B}$	-0.35	-0.45	$-\tilde{A} \succ -\tilde{B}$
Yu et al. (2013) [34]	$\alpha = 0$	751	1251	$\tilde{B} \succ \tilde{A}$	0.00133	0.0008	$-\tilde{A} \succ -\tilde{B}$
	$\alpha = 0.5$	1	1	$\tilde{A} \sim \tilde{B}$	1	1	$-\tilde{A} \sim -\tilde{B}$
	$\alpha = 1$	0.00133	0.0008	$\tilde{A} \succ \tilde{B}$	751	1251	$-\tilde{B} \succ -\tilde{A}$
Saneifard and Allahviranloo	(2012)[24]	0.62	0.56	$\tilde{A} \succ \tilde{B}$	0.37	0.32	$-\tilde{A} \succ -\tilde{B}$
The proposed method		(0.5, 0.015)	(0.5, 0.002)	$\tilde{A} \succ \tilde{B}$	$(-0.5, -0.015)$	$(-0.5, -0.002)$	$-\tilde{B} \succ -\tilde{A}$



(a) The type-1 fuzzy number $\tilde{A}$ and $\tilde{B}$

(b) The type-1 fuzzy number $\tilde{C}$ and $\tilde{C}'$

(c) The type-1 fuzzy number $\tilde{A}'$ and $\tilde{B}'$

(d) The type-1 fuzzy number $\tilde{A}''$ and $\tilde{B}''$

Fig. 5. The type-1 fuzzy numbers of Example 4.

Table 4
The comparative results of Example 4

Fuzzy numbers	$\tilde{A}$	$\tilde{B}$	$\tilde{A}'$	$\tilde{B}'$	$\tilde{A}''$	$\tilde{B}''$
Values	1.66	1.33	7.13	7.08	−0.16	−0.21
Ranking	$\tilde{A} \succ \tilde{B}$		$\tilde{A}' \succ \tilde{B}'$		$\tilde{A}'' \succ \tilde{B}''$	

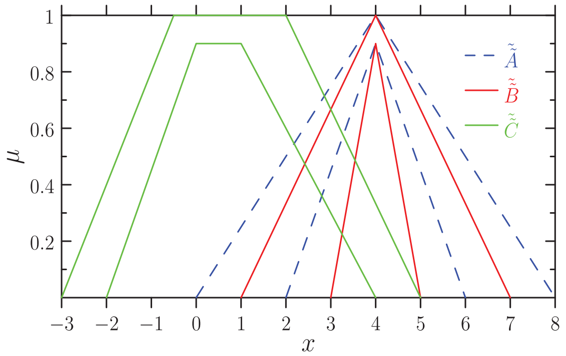


Fig. 6. The interval type-2 fuzzy numbers for Example 5.

Using the proposed method, the ranking of the two type-1 fuzzy numbers is $\tilde{A} \succ \tilde{B}$, $\tilde{A}' \succ \tilde{B}'$ and $\tilde{A}'' \succ \tilde{B}''$, respectively.

Table 4 lists the ranking results with the proposed method. From the comparative ranking results, it is right that the proposed method satisfies the property that the ranking order of $\tilde{A}$ and $\tilde{B}$ does not changed when adding any type-1 fuzzy numbers to them.

5.2. *The ranking for interval type-2 fuzzy numbers*

Example 5 is given to illustrate the properties of the proposed ranking method for interval type-2 fuzzy numbers.

Example 5. Consider the following interval type-2 fuzzy numbers, $\tilde{\tilde{A}} = [(4, 4, 4; 1), (4, 2, 2; 0.9)]$, $\tilde{\tilde{B}} = [(4, 3, 3; 1), (4, 1, 1; 0.9)]$, and $\tilde{\tilde{C}} = [(-1/2, 2, 5/2, 3; 1), (0, 1, 2, 3; 0.9)]$, which are shown in Fig. 6.

According to Equations (10), (12) and (11), Table 5 shows the possibility mean values and variation coefficient of these interval type-2 fuzzy numbers.

The ranking results for interval type-2 fuzzy numbers in Example 5 can be concluded as follows.

(1) From Table 5, it can easily be seen that the ranking result is $\tilde{\tilde{A}} \succ \tilde{\tilde{B}} \succ \tilde{\tilde{C}}$, which is consistent with peoples' intuition.
(2) For the images of interval type-2 fuzzy numbers $\tilde{\tilde{A}}$, $\tilde{\tilde{B}}$ and $\tilde{\tilde{C}}$ $(-\tilde{\tilde{A}}, -\tilde{\tilde{B}}, -\tilde{\tilde{C}})$, the ranking of them is $-\tilde{\tilde{C}} \succ -\tilde{\tilde{B}} \succ -\tilde{\tilde{A}}$, which is logical.
(3) Due to the fact that $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are symmetric interval type-2 fuzzy numbers with different spread in the bottom, the proposed ranking method can distinguish them clearly.

6. Conclusion

In this paper, we have firstly presented a new format of possibility mean and variation coefficient to rank type-1 fuzzy numbers. Compared with the currently main ranking methods, the proposed ranking method can avoid the defects that most ranking methods have, that is it not only correctly ranks the type-1 fuzzy numbers especially for the symmetric type-1 fuzzy numbers having different spans in the bottom, but also reasonably rank their images'. Meanwhile, some order properties are also introduced and discussed. Then, we extend the concept to interval type-2 fuzzy numbers environment, and introduced a new ranking method with possibility mean and variation coefficient, which not only discriminates the ranking result of any interval type-2 fuzzy numbers, but also the ranking of their images'. Finally, several examples are illustrated to compare the ranking with existing main ranking methods. It is demonstrated that the proposed ranking method provides a new alternative to rank type-1 fuzzy numbers and interval type-2 fuzzy numbers, and further enriches the ranking theories.

Table 5
The comparative results of Example 5

Fuzzy numbers	$\tilde{\tilde{A}}$	$\tilde{\tilde{B}}$	$\tilde{\tilde{C}}$	$-\tilde{\tilde{A}}$	$-\tilde{\tilde{B}}$	$-\tilde{\tilde{C}}$
Values	(3.62, 0.18)	(3.62, 0.07)	0.7	(−3.62, −0.18)	(−3.62, −0.07)	−0.7
Ranking	$\tilde{\tilde{A}} \succ \tilde{\tilde{B}} \succ \tilde{\tilde{C}}$			$-\tilde{\tilde{C}} \succ -\tilde{\tilde{B}} \succ -\tilde{\tilde{A}}$		

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